Renormalized solutions for stochastic transport equations and the regularization by bilinear multiplicative noise

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Abstract

A linear stochastic transport equation with non-regular coefficients is considered. Under the same assumption of the deterministic theory, all weak L^∞-solutions are renormalized. But then, if the noise is non-degenerate, uniqueness of weak L^∞-solutions does not require essential new assumptions, opposite to the deterministic case where for instance the divergence of the drift is asked to be bounded. The proof gives a new explanation why bilinear multiplicative noise may have a regularizing effect.

1 Introduction

Consider the deterministic linear transport equation in $\mathbb{R}^d$

$$\frac{\partial u}{\partial t} + (b \cdot \nabla)\, u = 0, \qquad u|_{t=0} = u_0 \tag{1}$$

in a non-regular framework, namely when the given vector field $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies

$$b, \operatorname{div} b \in L^1_{loc}\left([0,T] \times \mathbb{R}^d\right) \tag{2}$$

and the solution u is of class $L^\infty\left([0,T] \times \mathbb{R}^d\right)$, with $u_0 \in L^\infty\left(\mathbb{R}^d\right)$. Di Perna and Lions [10] have introduced the notion of renormalized solution to this equation: it is a solution such that

$$\frac{\partial \beta(u)}{\partial t} + (b \cdot \nabla)\, \beta(u) = 0 \tag{3}$$

for all functions $\beta \in C^1(\mathbb{R})$. When

$$b \in L^1\left(0, T; W^{1,1}_{loc}\left(\mathbb{R}^d\right)\right) \tag{4}$$

a basic commutator lemma between smoothing convolution and $(b \cdot \nabla)$ can be proved and, as a consequence, all L^∞-weak solutions are renormalized, see [10].

This fact is fundamental to prove uniqueness of weak solutions to equation (1). A main consequence is the uniqueness when the additional conditions

$$\frac{|b|}{1+|x|} \in L^1\left(0,T;L^\infty\left(\mathbb{R}^d\right)\right), \qquad \operatorname{div} b \in L^1\left(0,T;L^\infty\left(\mathbb{R}^d\right)\right)$$

are fulfilled, see [10]. These results have been generalized by Ambrosio [1] to BV_{loc}-vector fields (in place of $W^{1,1}_{loc}$). The BV-framework is the one adopted in the sequel, where we make extensive use of ideas and results from [1]. The notion of renormalized solutions has been investigated further by several authors, see for instance [3], [8], [9], [12], [18], [19], [21] and many others.

Many of the previous results can be extended quite easily to a stochastic framework of the form

$$du + (b\cdot\nabla)\,u dt + \sum_{k=1}^{d}\partial_k u \circ dW^k = 0, \qquad u|_{t=0} = u_0 \tag{5}$$

where W^k are independent Brownian motions; in particular, we give below the analogous result of renormalizability of all solutions, under the same assumptions on b as in [10]. But the reason for developing this extension is the fact that, after we have proved that all solutions are renormalized, we get uniqueness in cases not covered by the classical deterministic theory. One of our results is that, essentially, we may just get rid of the requirement $\operatorname{div} b\,(t,\cdot) \in L^\infty\left(\mathbb{R}^d\right)$ which is responsible for the exclusion of examples like $b\,(x) = \sqrt{|x|}$, $d = 1$:

Theorem 1 *If $b = b_1 + b_2$ with*

- $b, \operatorname{div} b \in L^1_{loc}([0,T]\times\mathbb{R}^d)$, $b_t \in BV_{loc}(\mathbb{R}^d;\mathbb{R}^d)$ *for a.e.* $t \in [0,T]$
- *For some* $N > d$
$$\int_0^T \int_{\mathbb{R}^d} \frac{|Db_t|}{(1+|x|)^N} dxdt < \infty$$
- $b_1 \in L^2\left(0,T;L^\infty\left(\mathbb{R}^d\right)\right)$
- $\frac{|b_2|}{1+|x|} \in L^1\left(0,T;L^\infty\left(\mathbb{R}^d\right)\right)$, $\operatorname{div} b_2 \in L^1\left(0,T;L^\infty\left(\mathbb{R}^d\right)\right)$

then there exists a unique weak L^∞-solution of equation (5).

One can accept a component b_1 of b which has no $L^\infty\left(\mathbb{R}^d\right)$-control on the divergence. We included the component b_2 in the statement to accept linear growth at infinity, but only with L^∞-divergence. In a sense, b_1 takes care of the irregular part of b in a bounded ball, b_2 of the more regular but possibly linear growth part of b at infinity.

That noise could improve the theory of transport equations was first discovered by [13]. The present work, being based on the same commutator lemma of the deterministic case, still requires the weak differentiability assumption (4).

On the contrary, the approach of [13] by stochastic characteristics allows one to get rid of the weak differentiability of b. In this sense the results of [13] are more advanced than the present ones. However, the assumptions here and in [13] are not directly comparable. The main condition assumed in [13] is

$$b \in L^\infty\left(0,T;C_b^\alpha\left(\mathbb{R}^d\right)\right)$$

together with a mild integrability of $\operatorname{div} b$. Here we may consider also discontinuous b in dimension $d>1$ (in dimension 1, assumption (4) implies continuity). To clarify, we give an example in section 6 which is covered here and not by [13]. The boundedness of b was also important in [13] to investigate the stochastic flow, while here it is easily removed. Moreover, the approach presented here generalizes more easily to space-dependent noise, but we do not stress this in this paper.

A part from the technical comparison of assumptions, one of the main purposes of this note is to describe a completely different reason, with respect to the one given in [13], that explains *why* noise improves the deterministic theory. In a sense, the reason explained here is more *structural*: it may hold true for equations possibly very different from linear transport ones, but having some common structural features. We know at present at least another example where it works, namely the system of infinitely many coupled equations

$$dX_n\left(t\right) = k_{n-1}X_{n-1}\left(t\right)\circ dW_{n-1}\left(t\right) - k_n X_{n+1}\left(t\right)\circ dW_n\left(t\right) \tag{6}$$

with $n\geq 1$, $X_0\left(t\right)=0$, $k_0=0$, and for instance $k_n=2^n$. See [7] for details. The proof in [7] has much in common with the one of the present paper, although at that time this structural fact was not identified.

In a few sentences, the reason why Stratonovich multiplicative noise, sometimes called Stratonovich *bilinear* noise, as that of equations (5) and (6), produces a regularization, is the following one. When we pass from Stratonovich to Itô form, a second order differential opertator A appears (see below its form for equation (5); think to a Laplacian in the easiest case):

$$du + \left(b\cdot\nabla\right)udt + \sum_{k=1}^{d}\partial_k u\circ dW^k = \frac{1}{2}\Delta u dt.$$

This equation is equivalent to (5), so there is no regularizing effect of Δ (it is fully compensated by the Itô term, as well understood in the theory of Zakai equation of filtering). A simple way to see that there is no regularization is to recall that the solution of (5) when b is smooth (see [17]) or like in [13] is given by

$$u\left(t,x\right) = u_0\left(\varphi_t^{-1}\left(x\right)\right)$$

for a properly defined stochastic flow φ_t of diffeomrphisms, so any irregularity of u_0 persists in time. But when we take expected value (assume the Itô term term is a martingale, thus with zero expected value) we get the parabolic equation

$$\frac{dE\left[u\right]}{dt} + \left(b\cdot\nabla\right)E\left[u\right] = \frac{1}{2}\Delta E\left[u\right].$$

Here we have a regularizing effect. The expected value $E\left[u\left(t,x\right)\right]$ is much more regular than $u\left(t,x\right)$.

Unfortunately we cannot use so easily this remark to prove uniqueness: if $u_0 = 0$, by the previous arguments we could only deduce $E\left[u\left(t,x\right)\right] = 0$ (this holds under more general assumptions than those of theorem 1), which does not imply $u = 0$.

But if we can prove that

$$d\beta\left(u\right) + \left(b\cdot\nabla\right)\beta\left(u\right)dt + \sum_{k=1}^{d}\partial_k\beta\left(u\right)\circ dW^k = 0$$

for all functions $\beta \in C^1\left(\mathbb{R}\right)$, then we pass to Itô form

$$d\beta\left(u\right) + \left(b\cdot\nabla\right)\beta\left(u\right)dt + \sum_{k=1}^{d}\partial_k\beta\left(u\right)dW^k = \frac{1}{2}\Delta\beta\left(u\right)$$

and take expectation

$$\frac{dE\left[\beta\left(u\right)\right]}{dt} + \left(b\cdot\nabla\right)E\left[\beta\left(u\right)\right] = \frac{1}{2}\Delta E\left[\beta\left(u\right)\right]. \tag{7}$$

Playing with positive functions β, this allows to prove $u = 0$. The advantage with respect to the deterministic case is that now we have the term $\Delta E\left[\beta\left(u\right)\right]$, which allows us to prove $E\left[\beta\left(u\right)\right] = 0$ under more general assumptions on b than for equation (3). At present, the weakeness of this method with respect to [13] is that we need to renormalize u.

An idea somewhat similar to this one was told to one of the author some time ago by B. Rozovskii, about a special variant of 3D Navier-Stokes equations. About this, unfortunately it is clear that one limitation of this approach is to linear equations, with deterministic coefficient b: the expected value would not commute in more general cases. Indeed, for nonlinear transport-like problems or linear with random b one can give counterexamples to a claim of regularization by noise, see [13] and [14]. But there are also positive nonlinear examples, of regularization by bilinear multiplicative noise, see [7], [15]. We are also aware of a work in progress by A. Debussche on a stochastic version of nonlinear Schrödinger equations, where a special multilicative noise has a regularizing effect that could be similar to what is described here. But each example requires special ad hoc arguments, at present. So the structural explanation of the present work is only a hint at the possibility that bilinear multiplicative noise regularizes, not a general fact.

Let us finally mention that, a posteriori, we notice similarities with the theory of stabilization by noise developed by Arnold, Crauel, Wihstutz, see [5], [4]. For a Stratonovich system written in astract fom as

$$dX_t = BX_t dt + \sum_k C_k X_t \circ dW_t^k$$

the Itô form is

$$dX_t = \left(B + \sum_k C_k^2 \right) X_t dt + \sum_k C_k X_t dW_t^k.$$

There are cases when C_k^2 is a "negative" operator (in a sense), like when $C_k^* = -C_k$ and $C_k C_k^*$ is positive definite. This is, in a sense, the case of the first order differential operators $C_k = \partial_k$. When C_k^2 are "negative", we may expect an increase of stability, becase again

$$\frac{d}{dt} E\left[X_t\right] = \left(B + \sum_k C_k^2 \right) E\left[X_t\right].$$

This is what has been proved in [5], [4], under suitable assumptions. At the PDE level, $\left(B + \sum_k C_k^2\right)$ may be regularizing, when B is not. However, going in more details, one can prove stabilization only when the trace of B is negative, see [5], [4], not in general as the operator $\left(B + \sum_k C_k^2\right)$ would suggest. This again shows that the simple argument about regularization of $E\left[X_t\right]$ (or $E\left[u\right]$ above) is only the signature of a possible but not sure regularization of the process itself.

2 Definitions and preliminaries

Consider the Stratonovich linear stochastic transport equation (5). To shorten some notation, highlight the structure and hint at more generality (not treated here), let us define a few differential operators. For a.e. $t \in [0, T]$, denote by $A_t, B_t, C_{t,k}$ the linear operators from $C_0^\infty\left(\mathbb{R}^d\right)$ to $L^1_{loc}\left(\mathbb{R}^d\right)$ defined as

$$\left(B_t f\right)(x) = \left(b\left(t,x\right) \cdot \nabla\right) f\left(x\right), \qquad \left(C_{t,k} f\right)(x) = \partial_k f\left(x\right)$$

$$\left(A_t f\right)(x) = \frac{1}{2} \sum_k C_{t,k} C_{t,k} f\left(x\right), \qquad f \in C_0^\infty\left(\mathbb{R}^d\right)$$

where, here, $A_t f = \Delta f$. Then denote by $A_t^*, B_t^*, C_{t,k}^*$ their formal adjoints, again linear operators from $C_0^\infty\left(\mathbb{R}^d\right)$ to $L^1_{loc}\left(\mathbb{R}^d\right)$, defined as

$$\left(B_t^* \varphi\right)(x) = -\left(b\left(t,x\right) \cdot \nabla\right) \varphi\left(x\right) - \varphi\left(x\right) \operatorname{div} b\left(t,x\right)$$
$$\left(C_{t,k}^* \varphi\right)(x) = -\partial_k \varphi\left(x\right)$$
$$\left(A_t^* \varphi\right)(x) = \sum_k C_{t,k}^* C_{t,k}^* \varphi\left(x\right), \qquad \varphi \in C_0^\infty\left(\mathbb{R}^d\right).$$

If $\varphi \in C_0^\infty\left(\mathbb{R}^d\right)$ we have $A_\cdot^* \varphi, B_\cdot^* \varphi, C_{\cdot,k}^* \varphi \in L^1_{loc}\left([0,T] \times \mathbb{R}^d\right)$. The next definition requires $b, \operatorname{div} b \in L^1\left(0, T; L^1_{loc}\left(\mathbb{R}^d\right)\right)$.

Definition 2 *If $u_0 \in L^\infty(\mathbb{R}^d)$, we say that a random field $u(t,x)$ is a weak L^∞-solution of equation (5) if $u \in L^\infty(\Omega \times [0,T] \times \mathbb{R}^d)$ and for all $\varphi \in C_0^\infty(\mathbb{R}^d)$ the real valued process $s \mapsto \int u_s C_{s,k}^* \varphi dx$ has a modification which is a continuous adapted semi-martingale and for all $t \in [0,T]$ we have P-a.s.*

$$\int u_t \varphi dx + \int_0^t \left(\int u_s B_s^* \varphi dx \right) ds + \sum_k \int_0^t \left(\int u_s C_{s,k}^* \varphi dx \right) \circ dW_s^k = \int u_0 \varphi dx.$$

A posteriori, form the equation itself, it follows that for all $\varphi \in C_0^\infty(\mathbb{R}^d)$ the real valued process $t \mapsto \int u_t \varphi dx$ has a continuous modification. We shall always use it when we write $\int u_t \varphi dx$, $\int u_t B_t^* \varphi dx$, $\int u_t C_{t,k}^* \varphi dx$.

The reason for the assumption that $\int u_s C_{s,k}^* \varphi dx$ is a continuous adapted semi-martingale is that the Stratonovich integrals

$$\int_0^t \left(\int u_s C_{s,k}^* \varphi dx \right) \circ dW_s^k$$

are thus well defined and equal to the corresponding Itô integrals plus half of the joint quadratic variation:

$$= \int_0^t \left(\int u_s C_{s,k}^* \varphi dx \right) dW_s^k + \frac{1}{2} \left[\int u_\cdot C_{\cdot,k}^* \varphi dx, W_\cdot^k \right]_t .$$

Recall, to help the intuition, that (with the notation $X_s = \int u_s C_{s,k}^* \varphi dx$)

$$\int_0^t X_s \circ dW_s^k = \lim_{n\to\infty} \sum_{t_i \in \pi_n, t_i \le t} \frac{X_{t_{i+1}\wedge t} + X_{t_i}}{2} \left(W_{t_{i+1}\wedge t} - W_{t_i} \right)$$

$$\int_0^t X_s dW_s^k = \lim_{n\to\infty} \sum_{t_i \in \pi_n, t_i \le t} X_{t_i} \left(W_{t_{i+1}\wedge t} - W_{t_i} \right)$$

$$\left[X_\cdot, W_\cdot^k \right]_t = \lim_{n\to\infty} \sum_{t_i \in \pi_n, t_i \le t} \left(X_{t_{i+1}\wedge t} - X_{t_i} \right) \left(W_{t_{i+1}\wedge t} - W_{t_i} \right)$$

where π_n is a sequence of finite partitions of $[0,T]$ with size $|\pi_n| \to 0$ and elements $0 = t_0 < t_1 < ...$, and the limits are in probability, uniformly in time on compact intervals. Details about these facts can be found in Kunita [17].

Proposition 3 *A weak L^∞-solution in the previous Stratonovich sense satisfies the Itô equation*

$$\int u_t \varphi dx + \int_0^t \left(\int u_s B_s^* \varphi dx \right) ds + \sum_k \int_0^t \left(\int u_s C_{s,k}^* \varphi dx \right) dW_s^k = \int u_0 \varphi dx + \int_0^t \left(\int u_s A_s^* \varphi dx \right) ds$$

for all $\varphi \in C_0^\infty(\mathbb{R}^d)$.

Proof. We have only to compute $\left[\int u_\cdot C^*_{\cdot,k}\varphi dx, W^k_\cdot\right]_t$. Notice that, by the equation itself,

$$\int u_t C^*_{t,k}\varphi dx+\int_0^t \left(\int u_s B^*_s C^*_{t,k}\varphi dx\right) ds+\sum_{k'}\int_0^t \left(\int u_s C^*_{s,k'} C^*_{t,k}\varphi dx\right)\circ dW^{k'}_s = \int u_0 C^*_{t,k}\varphi dx.$$

Thus, by classical rules, easily guessed by the Riemann sum approximations recalled above, we have

$$\left[\int u_\cdot C^*_{\cdot,k}\varphi dx, W^k_\cdot\right]_t = \int_0^t \left(\int u_s C^*_{s,k} C^*_{t,k}\varphi dx\right) ds.$$

The proof is complete, recalling the definition of A^*_t. ■

3 Renormalized solutions

Definition 4 *We say that a weak L^∞-solution of equation (5) is renormalized if for every $\beta \in C^1(\mathbb{R})$ the process $\beta(u(t,x))$ is a weak L^∞-solution of the same equation (5).*

Definition 5 *If $v_0 \in L^\infty(\mathbb{R}^d)$, we say that $v \in L^\infty([0,T]\times\mathbb{R}^d)$ is a weak L^∞-solution of the PDE*

$$\frac{\partial v}{\partial t} + b\cdot\nabla v = \frac{1}{2}Av, \qquad v|_{t=0} = v_0$$

if

$$\int v_t\varphi dx + \int_0^t \left(\int v_s B^*_s\varphi dx\right) ds = \int v_0\varphi dx + \int_0^t \left(\int v_s A^*_s\varphi dx\right) ds$$

for all $\varphi \in C_0^\infty(\mathbb{R}^d)$.

Definition 6 *Let M be a $n\times n$ matrix, and let $\theta \in C_c^\infty(\mathbb{R}^d)$ such that $\theta \geq 0$ and $\int\theta = 1$. Define*

$$\Lambda(M,\theta) := \int_{\mathbb{R}^d} |\langle Mz, \nabla\theta(z)\rangle|\, dz$$

and

$$I(\theta) := \int_{\mathbb{R}^d} |z||\nabla\theta(z)| dz$$

Theorem 7 *Suppose that b satisfies assumption (2), that, for a.e. $t \in [0,T]$, $b_t \in BV_{loc}(\mathbb{R}^d)$ and that, for every compact set $Q \subset \mathbb{R}^d$*

$$\int_0^T \int_Q |Db_t| dx dt < \infty$$

Denote with $D^s b$ and $D^a b$ the singular and absolutely continuous part of the measure Db respectively, and with M_t the rank one matrix of the polar decomposition $D^s b_t u = M_t |D^s b_t|$. Let $u \in L^\infty([0,T]\times\mathbb{R}^d)$ and $\theta \in C_c^\infty(\mathbb{R}^d)$ a smooth even nonnegative convolution kernel, such that $\operatorname{supp}\theta \subset B_1$. *Define* $\theta_\varepsilon(x) = \varepsilon^{-n}\theta(\frac{x}{\varepsilon})$, $L := \|u\|_{L^\infty([0,T]\times\mathbb{R}^d)}$ *and*

$$r_\varepsilon := b\cdot\nabla(u*\theta_\varepsilon) - (b\cdot\nabla u)*\theta_\varepsilon$$

Then, for every compact set $Q \subset \mathbb{R}^d$

$$\limsup_{\varepsilon\downarrow 0}\int_0^T\int_Q |r_\varepsilon|dxdt \le LI(\theta)|D^s b|([0,T]\times Q) \tag{8}$$

and

$$\limsup_{\varepsilon\downarrow 0}\int_0^T\int_Q |r_\varepsilon|dxdt \le L\int_0^T\int_Q \Lambda(M_t(x),\theta)d|Db^s|(t,x) + L(d+I(\theta))|D^a b|([0,T]\times Q) \tag{9}$$

Moreover for every $\delta > 0$ and vectors η and ζ, θ can be choosen such that: $\Lambda(\eta\otimes\zeta,\theta) < \delta$.

In the sequel we will need, in addition to the estimate on $\limsup_{\varepsilon\to 0}\|r_\varepsilon\|_{L^1}(B_R)$ given by theorem 7, an estimate on $\|r_\varepsilon\|_{L^1}$. Therefore the following proposition will be useful.

Proposition 8 *Suppose that* $u \in L^\infty(\mathbb{R}^d)$, $b \in BV_{loc}(\mathbb{R}^d;\mathbb{R}^d)$, $\operatorname{div} b \in L^1_{loc}(\mathbb{R}^d)$ *and* $\theta \in C_c^\infty(\mathbb{R}^d)$ *is a smooth even nonnegative convolution kernel, such that* $\operatorname{supp}\theta \subset B_1$. *Then, exists an even convolution kernel ρ, with* $\operatorname{supp}\rho \subset B_1$ *such that, for every measurable φ, it holds:*

$$\int |r_\varepsilon\varphi|dx \le C_\theta\|u\|_{L^\infty((\operatorname{supp}\varphi)_\varepsilon)}|Db|(|\varphi|*\rho_\varepsilon)$$

where $(\operatorname{supp}\varphi)_\varepsilon = \{x \in \mathbb{R}^d : \operatorname{dist}(x,\operatorname{supp}\varphi) \le \varepsilon\}$ *Therefore, for a.e. every $x \in \mathbb{R}^d$ it holds:*

$$|r_\varepsilon|(x) \le C_\theta\|u\|_{L^\infty(B(x,\varepsilon))}(|Db|*\rho_\varepsilon)(x)$$

Proof. First of all note that the second inequality is an easy consequence of the first one. From the definition of r_ε it follows:

$$\int |r_\varepsilon\varphi|dx \le \int\int |\varphi(x)u(y)\left[\theta_\varepsilon(x-y)\operatorname{div} b(y) + \nabla_y\theta_\varepsilon(x-y)\cdot(b(y)-b(x))\right]|\,dydx$$

$$\le \int |u(y)\operatorname{div} b(y)\,(\varphi*\theta_\varepsilon)(y)|\,dy + \int|\varphi(x)|\int|u(x+\varepsilon z)|\left|\frac{b(x+\varepsilon z)-b(x)}{\varepsilon}\cdot\nabla\theta(-z)\right|dzdx$$

Note that

$$\int |u(y)\operatorname{div} b(y)\,(\varphi*\theta_\varepsilon)(y)|\,dy \le d\|u\|_{L^\infty((\operatorname{supp}\varphi)_\varepsilon)}|Db|(|\varphi*\theta_\varepsilon|)$$

and that

$$\int |\varphi(x)| \int |u(x+\varepsilon z)| \left| \frac{b(x+\varepsilon z)-b(x)}{\varepsilon} \cdot \nabla\theta(-z) \right| dz dx$$

$$= \int_{\mathbb{R}^d} |\varphi(x)| \int_{\mathbb{R}^d} |u(x+\varepsilon z)| \left| \int_{\mathbb{R}} Db(x+tz)(z) \cdot \nabla\theta(-z) \left(\frac{1}{\varepsilon} 1_{[-\varepsilon,0]}(-t) \right) \right| dt dz dx$$

$$\leq \int \int \int |\varphi(y-tz)||u(y-(\varepsilon-t)z)||Db(y)||z||\nabla\theta(-z)| \left(\frac{1}{\varepsilon} 1_{[-\varepsilon,0]}(-t) \right) dt dz dy$$

Since $\operatorname{supp}\theta \subset B_1$, with the change of variable $r = zt$, we obtain:

$$\int \int |z||\nabla\theta(-z)||\varphi(y-tz)| \left(\frac{1}{\varepsilon} 1_{[-\varepsilon,0]}(-t) \right) dt dz$$

$$\leq \|\nabla\theta\|_\infty \frac{1}{\varepsilon} \int_0^\varepsilon \frac{1}{t^d} \int_{r \in B(0,t)} |\varphi(y-r)| dr dt = C_\theta |\varphi| * \rho_\varepsilon^{'}(y)$$

where $\rho_\varepsilon^{'}(z) = \frac{1}{\varepsilon} \int_0^\varepsilon \frac{1}{t^d} 1_{|z| \leq t} dt$ is (up to a constant independent of ε) an L^1 convolution kernel, with support contained in B_ε. Therefore we have proved

$$\int |\varphi(x)| \int |u(x+\varepsilon z)| \left| \frac{b(x+\varepsilon z)-b(x)}{\varepsilon} \cdot \nabla\theta(-z) \right| dz dx \leq C_\theta \|u\|_{L^\infty((\operatorname{supp}\varphi)_\varepsilon)} |Db|(|\varphi| * \rho_\varepsilon^{'})$$

So, defining $\rho_\varepsilon = \frac{\theta_\varepsilon + \rho_\varepsilon^{'}}{2}$ the proof is complete. ∎

Theorem 9 *Suppose that b satisfies assumption (2), that, for a.e. $t \in [0,T]$, $b_t \in BV_{loc}(\mathbb{R}^d)$ and that, for every compact set $Q \subset \mathbb{R}^d$*

$$\int_0^T \int_Q |Db_t| dx dt < \infty$$

Then all weak L^∞-solution are renormalized and, for any given $\beta \in C^1(\mathbb{R})$, the function

$$v(t,x) = E[\beta(u(t,x))]$$

is a weak L^∞-solution of the equation

$$\frac{\partial v}{\partial t} + b \cdot \nabla v = \frac{1}{2} A v, \qquad v|_{t=0} = \beta(u_0).$$

Proof. Step 1 Let u be a weak L^∞ solution of equation (5). Let $\theta \in C_c^\infty(\mathbb{R}^d)$ be a even smooth convolution kernel, and define $\theta_\varepsilon(x) := \frac{1}{\varepsilon^d}\theta(\frac{x}{\varepsilon})$ and $u_t^\varepsilon = u * \theta_\varepsilon$. Fix $y \in \mathbb{R}^d$, and consider the test function $\varphi(\cdot) = \theta_\varepsilon(y - \cdot)$. From the definition of week L^∞ solution, we have:

$$u_t^\varepsilon(y) - \int_0^t (u_s \operatorname{div} b_s) * \theta_\varepsilon(y) + (u_t b_t) * \nabla\theta_\varepsilon(y) ds + \sum_{k=1}^d \int_0^t D_k u_s^\varepsilon(y) \circ dW_s^k = u_0^\varepsilon(y)$$

Therefore, differentiating and multiplying for $\beta^{'}(u_t^\varepsilon)$ it holds a.s. in the sense of the distributions on $[0,T]\times\mathbb{R}^d$,

$$\frac{d}{dt}\beta(u_t^\varepsilon)(y)+b(y)\cdot\nabla\beta(u_t^\varepsilon)(y)+\beta^{'}(u_t^\varepsilon)(y)r_t^\varepsilon(y)+\sum_{k=1}^{d}D_k\beta^{'}(u_t^\varepsilon)(y)\circ dW_s^k=0$$

where $r_t^\varepsilon := (b_t\cdot\nabla u_t)*\theta_\varepsilon - b_t\cdot\nabla(u_t*\theta_\varepsilon)\in L^1_{loc}([0,T]\times\mathbb{R}^d)$. So, for any $\varphi\in C_c^\infty(\mathbb{R}^d)$ we have

$$\begin{aligned}\int\beta(u_t^\varepsilon)\varphi(x)dx-\int_0^t\left(\int\beta(u_s^\varepsilon)\left[\operatorname{div}b_s\varphi+b_s\cdot\nabla\varphi\right]dx\right)ds\\-\sum_{k=1}^{d}\int_0^t\int\beta(u_s^\varepsilon)D_k\varphi dx\circ dW_s^k-\int\beta(u_0^\varepsilon)\varphi dx=-\int_0^t\int\varphi\beta^{'}(u_s^\varepsilon)r_s^\varepsilon dxds\end{aligned}\tag{10}$$

From the definition $u^\varepsilon := u*\theta_\varepsilon$ it follows $\beta(u_t^\varepsilon)\to\beta(u_t)$ in $L^p(\Omega\times[0,T]\times B_R)$ for every $p\in[1,\infty)$ and every $R>0$. Moreover $\beta(u_t^\varepsilon)\to\beta(u_t)$ for a.e. (ω,t,x). Therefore, for any sequence $\varepsilon_n\to 0$ it is possible to extract a subsequence still denoted by ε_n, such that for a.e. $\omega\in\Omega$ the left hand side converge to

$$\int\beta(u_t)\varphi(x)dx-\int_0^t\left(\int\beta(u_s)\left[\operatorname{div}b_s\varphi+b_s\cdot\nabla\varphi\right]dx\right)ds-\sum_{k=1}^{d}\int_0^t\int\beta(u_s)D_k\varphi dx\circ dW_s^k-\int\beta(u_0)\varphi dx$$

Therefore for a.e. $\omega\in\Omega$, $\beta^{'}(u_t^{\varepsilon_n})r_t^{\varepsilon_n}$, which is uniformly bounded in $L^1([0,T]\times B_R)$, converge to a signed measure σ with finite total variation on $[0,T]\times B_R$. So, to show that u is a renormalized solution it is sufficient to show that a.s. $\sigma=0$ on $[0,T]\times B_R$. Note that the limit of the left hand side of equation (10) does not depend on the choice of θ and therefore σ does not depend on the choice of θ. Thanks to the first estimate of theorem 7, and to the boundness of $\beta^{'}(u^{\varepsilon_n})$, σ is a.s. singular with respect to the $d+1$ dimensional Lebesgue measure. Moreover thanks to the second estimate of theorem 7, and to the fact that σ is singular, we have the estimate:

$$|\sigma|\le\|\beta^{'}(u)\|_\infty\|u\|_\infty\Lambda(M_t(x),\theta)d|D b^s|(t,x)$$

Let g be the Radon-Nykodym derivative of σ with respect to $|D^s b|$. It holds, for every smooth even nonnegative convolution kernel θ, $g\le\|\beta^{'}(u)\|_\infty\|u\|_\infty\Lambda(M_t,\theta)$, $|D^s b|$-a.e. Let $D\subset C_c^\infty(B_1)$ be a countable set, dense with respect to the norm $W^{1,1}(B_1)$, in the set:

$$R:=\left\{\theta\in W^{1,1}(B_1):\theta\ge 0,\,\theta(x)=\theta(-x)\,\forall x\in\mathbb{R}^d,\,\int\theta=1\right\}$$

Being D countable it holds $g(t,x)\le\|\beta^{'}(u)\|_\infty\|u\|_\infty\inf_{\theta\in D}\Lambda(M_t(x),\theta)$ for $|D^s b|$-a.e. (t,x) and being D dense it holds also

$$g(t,x)\le\|\beta^{'}(u)\|_\infty\|u\|_\infty\inf_{\theta\in R}\Lambda(M_t,\theta)$$

for $|D^s b|$-a.e. (t,x) Thanks to Alberti rank one theorem we know that M_t has rank one, and so $g=0$ and $|\sigma|=0$.
Step 2. Thanks to the previous step it holds a.s. and for every $t\in[0,T]$ and $\varphi\in C_c^\infty(\mathbb{R}^d)$

$$\begin{aligned}\int\beta(u_t)\varphi(x)dx-\int_0^t\left(\int\beta(u_s)\left[\operatorname{div}b_s\varphi+b_s\cdot\nabla\varphi\right]dx\right)ds\\-\sum_{k=1}^d\int_0^t\int\beta(u_s)D_k\varphi dx\circ dW_s^k-\int\beta(u_0)\varphi dx=0\end{aligned}\tag{11}$$

Applying proposition 3 and taking the mean value we obtain that $v=E[\beta(u)]$ satisfies

$$\int v_t\varphi(x)dx-\int_0^t\left(\int v_s\left[\operatorname{div}b_s\varphi+b_s\cdot\nabla\varphi\right]dx\right)ds-\frac{1}{2}\int_0^t\int v_s\Delta\varphi dxds-\int v_0\varphi dx=0\tag{12}$$

The proof is complete. ■

4 Proof of theorem 1

Notice that only here the strict ellipticity of the operator Δ is used (or the non-degeneracy assumption of the coefficients $\left(\sigma^k\right)$ in the last section), since we need parabolic regularization to prove uniqueness without the assumption $\operatorname{div}b\,(t,\cdot)\in L^\infty\left(\mathbb{R}^d\right)$.

Let us make more precise a detail that was not said in the introduction. When we say that two weak L^∞-solutions coincide, we mean they are in the same equivalence class of $L^\infty\left(\Omega\times[0,T]\times\mathbb{R}^d\right)$. It follows that, for every $\varphi\in C_0^\infty\left(\mathbb{R}^d\right)$, the continuous processes $\int u_t\varphi dx$ of definition 2 are indistinguishable. Let us split the proof in a few steps.

Step 1 (from the SPDE to a parabolic PDE). Call u the difference of two solutions. It is a weak L^∞-solution with zero initial condition. By theorem 9, u is renormalized and, given $\beta_0\in C^1$, the function

$$v\,(t,x)=E\left[\beta_0\left(u\,(t,x)\right)\right]$$

is a weak L^∞ solution of the equation

$$\frac{\partial v}{\partial t}+b\cdot\nabla v=\frac{1}{2}\Delta v.$$

Choose β_0 such that $\beta_0\,(0)=0$, so $v|_{t=0}=0$, and $\beta_0\,(u)>0$ for $u\neq0$. If we prove that $v_t=0$, we have proved $u_t=0$, P-a.s. This easily implies that u is the zero element of $L^\infty\left(\Omega\times[0,T]\times\mathbb{R}^d\right)$, which is our claim.

Step 2. (uniqueness for the parabolic equation). Define $v_\varepsilon=E[\beta_0(u^\varepsilon)]$ and $r_t^\varepsilon:=(b_t\cdot\nabla u_t)*\theta_\varepsilon-b_t\cdot\nabla(u_t*\theta_\varepsilon)$. From the proof of the previous theorem we know that $v_\varepsilon\to v$ a.e. and in $L^p_{loc}([0,T]\times B_R)$ for every $p\in[1,\infty)$ and every

$R>0$. Therefore, to prove $v=0$, it is sufficient to prove that, $\int \varphi v_\varepsilon^2 dx \to 0$ for a smooth and positive function φ. We will consider the function

$$\varphi(x) = (1+|x|)^{-N}$$

where $N>d$ is the number given in the assumptions of the theorem. Note that it holds

$$\nabla\varphi(x) = -N(1+|x|)^{-N-1}\frac{x}{|x|}$$

hence

$$(1+|x|)\,|\nabla\varphi(x)| \le N\,|\varphi(x)|$$

From identity (10) of the previous theorem, using proposition 3, taking the mean value and then differentiating and multiplying by $2v^\varepsilon$, we have

$$\frac{d}{dt}\int \varphi\,|v_\varepsilon|^2 = -2\int \varphi v_\varepsilon b\cdot\nabla v_\varepsilon + \int \varphi v_\varepsilon \Delta v_\varepsilon - 2\int \varphi v_\varepsilon E\left[\beta_0^{'}(u_t^\varepsilon) r_t^\varepsilon\right] \qquad (13)$$

for every $\varphi \in C_c^\infty(\mathbb{R}^d)$. Using the boundedness of $v_\varepsilon, \nabla v_\varepsilon, \Delta v_\varepsilon$ and the integrability over $\mathbb{R}^d$ of φ (see also the next step for the finiteness of the term $\int \varphi v_\varepsilon E\left[\beta_0^{'}(u_t^\varepsilon) r_t^\varepsilon\right]$) it is easy to see that equation (13) holds for $\varphi(x) = (1+|x|)^{-N}$. Moreover

$$\begin{aligned}\int \varphi v_\varepsilon \Delta v_\varepsilon &= -\int \varphi\,|\nabla v_\varepsilon|^2 - \int v_\varepsilon \nabla\varphi\cdot\nabla v_\varepsilon \le -\int \varphi\,|\nabla v_\varepsilon|^2 + N\int |v_\varepsilon|\,|\varphi|\,|\nabla v_\varepsilon| \\ &\le -\frac{1}{2}\int \varphi\,|\nabla v_\varepsilon|^2 + \frac{N^2}{2}\int |v_\varepsilon|^2\,|\varphi|\end{aligned}$$

$$\int \varphi v_\varepsilon b\cdot\nabla v_\varepsilon = \int \varphi v_\varepsilon b_1\cdot\nabla v_\varepsilon + \int \varphi v_\varepsilon b_2\cdot\nabla v_\varepsilon$$

$$-2\int \varphi v_\varepsilon b_1\cdot\nabla v_\varepsilon \le 2\,\|b_1(t)\|_{L^\infty(\mathbb{R}^d)}\int \varphi\,|v_\varepsilon|\,|\nabla v_\varepsilon| \le \frac{1}{4}\int \varphi\,|\nabla v_\varepsilon|^2 + C\,\|b_1(t)\|^2_{L^\infty(\mathbb{R}^d)}\int \varphi\,|v_\varepsilon|^2\,dx$$

$$\begin{aligned}-2\int \varphi v_\varepsilon b_2\cdot\nabla v_\varepsilon &= -\int \varphi b_2\cdot\nabla v_\varepsilon^2 = \int v_\varepsilon^2 b_2\cdot\nabla\varphi + \int v_\varepsilon^2\varphi\,\mathrm{div}\,b_2 \\ &\le \left\|\frac{b_2(t)}{1+|x|}\right\|^2_{L^\infty(\mathbb{R}^d)}\int v_\varepsilon^2\,(1+|x|)\,|\nabla\varphi(x)|\,dx + \|\mathrm{div}\,b_t\|_{L^\infty(\mathbb{R}^d)}\int v_\varepsilon^2\varphi dx \\ &\le \left(\left\|\frac{b_2(t)}{1+|x|}\right\|^2_{L^\infty(\mathbb{R}^d)} + \|\mathrm{div}\,b_t\|_{L^\infty(\mathbb{R}^d)}\right) N\int v_\varepsilon^2\varphi dx.\end{aligned}$$

Summarizing,

$$\frac{d}{dt}\int \varphi\,|v_\varepsilon|^2 + \frac{1}{4}\int \varphi\,|\nabla v_\varepsilon|^2 \le C_N\alpha(t)\int |v_\varepsilon|^2\varphi dx + C\,\|v\|_{L^\infty([0,T]\times\mathbb{R}^d)}\int \varphi E\left[\left|\beta_0^{'}(u_t^\varepsilon) r_t^\varepsilon\right|\right] dx$$

where

$$\alpha(t) := \|b_1(t)\|^2_{L^\infty(\mathbb{R}^d)} + \left\| \frac{b_2(t)}{1+|x|} \right\|^2_{L^\infty(\mathbb{R}^d)} + \|\mathrm{div}\, b_t\|_{L^\infty(\mathbb{R}^d)}$$

is integrable. By Gronwall lemma and the result of the next step we deduce

$$\lim_{\varepsilon\to 0} \int \varphi(x) |v_\varepsilon(t,x)|^2 dx = 0$$

for all $t \in [0,T]$, and thus $v = 0$.

Step 3. It remains to show that

$$\int_0^T \int (1+|x|)^{-N} E\left[\left|\beta^{'}(u_t^\varepsilon) r_t^\varepsilon\right|\right] dxdt \to 0$$

First of all note that, given a convolution kernel ρ_ε, for ε sufficiently small it holds $\frac{1}{(1+|x|)^N} * \rho_\varepsilon \le 2\frac{1}{(1+|x|)^N}$. Therefore we can apply proposition 8 and we obtain:

$$\int_0^T \int_{|x|\ge R} (1+|x|)^{-N} |\beta^{'}(u^\varepsilon) r_\varepsilon| dxdt \le C\|\beta^{'}(u)\|_\infty \|u\|_\infty \int_0^T \int_{|x|\ge R} \frac{|Db_t|}{(1+|x|)^N} dxdt < \infty \tag{14}$$

Since

$$\lim_{R\to+\infty} \int_0^T \int_{|x|\ge R-1} \frac{|Db|}{(1+|x|)^N} dxdt = 0$$

it is sufficient to prove that, for every $R > 0$

$$\lim_{\varepsilon\to 0} \int_0^T \int_{B_R} E\left[\left|\beta^{'}(u_t^\varepsilon) r_t^\varepsilon\right|\right] dxdt = 0$$

Thanks to the bound $\int_0^T \int_{B_R} \left|\beta^{'}(u_t^\varepsilon) r_t^\varepsilon\right| dxdt \le C\|\beta^{'}(u)\|_\infty \|u\|_\infty \int_0^T \int_{B_{R+1}} |Db_t| dxdt$, which holds a.s. for proposition 8, is sufficient to prove that for any sequence ε_n exists a subsequence, still denoted by ε_n such that a.s. $\int_0^T \int_{B_R} \left|\beta^{'}(u_t^\varepsilon) r_t^\varepsilon\right| dxdt \to 0$. This follows from the proof of the previous theorem, where it was proved that, for every sequence ε_n there exist a subsequence still denoted by ε_n such that $\beta^{'}(u_t^{\varepsilon_n}) r_t^{\varepsilon_n}$ a.s. converges to a measure σ, with finite total variation on $[0,T] \times B_R$, and then it was proved that a.s. $\sigma = 0$.

5 Remarks on a few variants

In a sentence, the core of the method is the commutator lemma (or renormalizability of solutions) which requires classical assumptions on (b,u), plus a theorem of uniqueness of non-negative L^∞-solutions $v = E[\beta(u)]$ of the parabolic equation (7). There are recent advanced results on this parabolic equation under assumption on b coherent with the present framework, expecially [12] and

[19]. But they do not fit precisely with our purposes for different reasons. For instance, [19] show that, due to the Laplacian, one can weaken the assuptions on b, but the uniqueness is in the class of solutions v with the usual variational regularity (including $L^2\left(0,T;W^{1,2}\left(\mathbb{R}^d\right)\right)$). We do not know that $E\left[\beta\left(u\right)\right]$ has this regularity. The paper [12] deals with only L^∞-solutions v of the parabolic equation (7) and proves uniqueness, but again assuming $\operatorname{div} b \in L^\infty$, the generalization of which is one of our main purposes, otherwise the theory would be equal to the deterministic one.

This is the reason why we have given above a self-contained proof of uniqueness for equation (7). There are other proofs, under easier or different assumptions. We have given above, as a main theorem, the one which allows us to deal with the BV set-up and linear growth b. In other directions of generality, or simplicity of proofs, we have the following two results. We only sketch the proofs. The first theorem is a particular case of theorem 1, with $b = b_1$. We give a very simple proof by semigroup theory, which could be generalized to other analytic semigroups in $L^1\left(\mathbb{R}^d\right)$ different from the heat semigroup.

Theorem 10 *If, in addition to hypothesis (2), we assume*

$$Db \in L^1\left([0,T]\times\mathbb{R}^d\right), \qquad b \in L^\infty\left([0,T]\times\mathbb{R}^d\right).$$

(which includes (4)), then there exists a unique weak L^∞-solution of equation (5).

Proof. From the SPDE to the parabolic PDE (7) the proof is the same as in theorem 1. We have only to show uniqueness of the solution v to (7). Let θ_ε be the mollifiers introduced above and let $v_\varepsilon\left(t,\cdot\right) = \theta_\varepsilon * v\left(t,\cdot\right)$ (convolution in space). Take β as in the proof of theorem 1. We have

$$\frac{\partial v_\varepsilon}{\partial t} + b\cdot\nabla v_\varepsilon = \frac{1}{2}\Delta v_\varepsilon + r_\varepsilon, \qquad v_\varepsilon\left(0\right) = 0$$

where r_ε is the usual commutator. Therefore

$$v_\varepsilon\left(t\right) = \int_0^t T_{t-s}\left(r_\varepsilon\left(s\right) - b\left(s\right)\cdot\nabla v_\varepsilon\left(s\right)\right)ds$$

where

$$\left(T_t f\right)\left(x\right) := \int_{\mathbb{R}^d} f\left(x+y\right)\left(2\pi t\right)^{-d/2}\exp\left(-\frac{\left|y\right|^2}{2t}\right)dy.$$

Notice that $v_\varepsilon \in L^\infty\left([0,T]\times\mathbb{R}^d\right)$, hence all the previous integrals are well defined.

We have

$$\left\|r_\varepsilon\left(t\right)\right\|_{L^1(\mathbb{R}^d)} \leq C\left\|v\left(t\right)\right\|_{L^\infty(\mathbb{R}^d)}\left\|b\left(t\right)\right\|_{W^{1,1}(\mathbb{R}^d)}$$

and $\left\|r_\varepsilon\left(t\right)\right\|_{L^1(\mathbb{R}^d)} \to 0$ as $\varepsilon \to 0$, see [20], lemma 2.3. Moreover, the heat semigroup has the following property:

$$\left\|\int_0^t T_{t-s}f\left(s\right)ds\right\|_{W^{1,1}(\mathbb{R}^d)} \leq \int_0^t \frac{C}{\left(t-s\right)^{1/2}}\left\|f\left(s\right)\right\|_{L^1(\mathbb{R}^d)}ds.$$

This implies, with $f(s) = r_\varepsilon(s) - b(s) \cdot \nabla v_\varepsilon(s)$

$$\int_0^T \|v_\varepsilon(t)\|_{W^{1,1}(\mathbb{R}^d)} dt \le \int_0^T \int_0^t \frac{C}{(t-s)^{1/2}} \|f(s)\|_{L^1(\mathbb{R}^d)} ds dt$$
$$\le C\sqrt{T} \int_0^T \|f(s)\|_{L^1(\mathbb{R}^d)} ds$$

and thus

$$\int_0^T \|v_\varepsilon(t)\|_{W^{1,1}(\mathbb{R}^d)} dt \le C\sqrt{T} \int_0^T \|r_\varepsilon(t)\|_{L^1(\mathbb{R}^d)} dt + C\sqrt{T} \int_0^T \|b\|_{L^\infty} \|v_\varepsilon(t)\|_{W^{1,1}(\mathbb{R}^d)} dt.$$

For small $T > 0$ this gives us

$$\int_0^T \|v_\varepsilon(t)\|_{W^{1,1}(\mathbb{R}^d)} dt \le C_T \int_0^T \|r_\varepsilon(t)\|_{L^1(\mathbb{R}^d)} dt$$

and thus (by the properties of r_ε recalled above and Lebesgue theorem) $\lim_{\varepsilon\to 0} \int_0^T \|v_\varepsilon(t)\|_{W^{1,1}(\mathbb{R}^d)} dt = 0$. This implies $v = 0$. The proof is complete. ■

Next theorem is a little generalization of theorem 1 in the direction of the so called Prodi-Serrin condition of fluid dynamics. The stronger condition $\frac{2}{q} + \frac{d}{p} < 1$ is the basic one in the work [16]. Under the same assumption it has been proved in [11] that the stochastic characteristics generate a flow of Hölder homeomorphisms, so probably this assumption could be related to a future generalization of the approach of [13].

Theorem 11 *Theorem 1 remains true if we replace assumption $b_1 \in L^2(0, T; L^\infty(\mathbb{R}^d))$ with*

$$b \in L^q(0, T; L^p(\mathbb{R}^d)), \qquad \frac{2}{q} + \frac{d}{p} \le 1, \qquad p, q \in (1, \infty).$$

Proof. We have only to modify the estimate for $\int \varphi v_\varepsilon b_1 \cdot \nabla v_\varepsilon$. Here we use the following bounds:

$$\left| \int \varphi v_\varepsilon b_1 \cdot \nabla v_\varepsilon dx \right| \le \frac{1}{4} \int \varphi |\nabla v_\varepsilon|^2 dx + C^* \int \varphi |b_1|^2 v_\varepsilon^2 dx$$

$$\int \varphi |b_1|^2 v_\varepsilon^2 dx \le \left(\int |b_1|^p dx \right)^{2/p} \left(\int |\sqrt{\varphi} v_\varepsilon|^{2p/(p-2)} dx \right)^{\frac{p-2}{p}}$$

and by an interpolation inequality

$$\le C \left(\int |b_1|^p dx \right)^{2/p} \|\sqrt{\varphi} v_\varepsilon\|_{W^{\gamma,2}}^2$$
$$\le C \left(\int |b_1|^p dx \right)^{2/p} \|\sqrt{\varphi} v_\varepsilon\|_{L^2}^{2-2\gamma} \|\sqrt{\varphi} v_\varepsilon\|_{W^{1,2}}^{2\gamma}$$
$$\le \frac{1}{8C^*} \|\sqrt{\varphi} v_\varepsilon\|_{W^{1,2}}^2 + C \left(\int |b_1|^p dx \right)^{\frac{2}{p} \cdot \frac{1}{1-\gamma}} \|\sqrt{\varphi} v_\varepsilon\|_{L^2}^2$$

where $\gamma < 1$, $\frac{p-2}{2p} = \frac{1}{2} - \frac{\gamma}{d}$, namely $\gamma = \frac{d}{p}$. We have

$$\|\sqrt{\varphi} v_\varepsilon\|_{W^{1,2}}^2 \leq \int \varphi v_\varepsilon^2 dx + 2\int \left|v_\varepsilon \nabla\sqrt{\varphi}\right|^2 dx + 2\int \varphi \left|\nabla v_\varepsilon\right|^2 dx$$

and, recalling that $|\nabla\varphi| \leq N|\varphi|$ and $|\varphi| \leq 1$,

$$\int \left|v_\varepsilon \nabla\sqrt{\varphi}\right|^2 dx \leq \frac{1}{2}\int v_\varepsilon^2 \frac{|\nabla\varphi|^2}{\sqrt{\varphi}} dx \leq \frac{1}{2}\int v_\varepsilon^2 \frac{N^2|\varphi|^2}{\sqrt{\varphi}} dx \leq \frac{N^2}{2}\int v_\varepsilon^2 \varphi dx.$$

Therefore

$$\|\sqrt{\varphi} v_\varepsilon\|_{W^{1,2}}^2 \leq C_N \int v_\varepsilon^2 \varphi dx + 2\int \varphi \left|\nabla v_\varepsilon\right|^2 dx.$$

Summarizing, we have proved that

$$\left|\int \varphi v_\varepsilon b_1 \cdot \nabla v_\varepsilon dx\right| \leq \frac{1}{4}\int \varphi \left|\nabla v_\varepsilon\right|^2 dx + C_N\left(1 + \|b_1\|_{L^p(\mathbb{R}^d)}^{\frac{2}{1-\gamma}}\right)\int v_\varepsilon^2 \varphi dx$$

for a suitable constant C_N. It is now easy to complete the proof of theorem 1 by Gronwall lemma, if we check that

$$\frac{2}{1-\gamma} \leq q.$$

Since $\gamma = \frac{d}{p}$, the inequality is $\frac{2}{q} \leq 1-\gamma = 1-\frac{d}{p}$, which is preisely our assumption. The proof is complete. ■

6 Example

We give a simple example, with the flavour of shear flows, which is covered by theorem 1 but apparently not by the previous deterministic or stochastic works.

Consider $d = 2$ and

$$b(x,y) = sign(y)\begin{pmatrix} 1 \\ 2\sqrt{|y|} \end{pmatrix}.$$

This is not covered by [10] or [1] because

$$\operatorname{div} b = \frac{1}{\sqrt{|y|}}$$

is not bounded, and not by [13] because b is discontinuous. Without noise, the equations of characteristics

$$X' = sign(Y)$$
$$Y' = 2sign(Y)\sqrt{|Y|}$$

have multiple solutions from every initial condition on the line $(x_0, 0)$, $x_0 \in \mathbb{R}$, the ideal surface of separation of this "compressible shear flow", which move away fron the surface as in a sort of instability process. Using these multiple solutions one can write down multiple solutions of the deterministic trasport equation, with any initial condition u_0. On the contrary, the stochastic equation (5) is well posed, since we may apply theorem 1 with

$$b_1(x, y) = sign(y) \left(\frac{1}{2\sqrt{|y| \wedge 1}} \right)$$

and $b_2 = b - b_1$. We have $\partial_y b^{(1)}(x, y) = 2\delta(y)$, integrable with $N = 2$ in the sense of the assumptions of theorem 1; and Db_2 bounded, so it is easy to see that the assumptuions of the main theorem are fulfilled and we have uniqueness.

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